South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, Proceedings (2022), pp. 115-138

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

FURTHER RESULTS ON DIVISOR CORDIAL GRAPH

C. M. Barasara and Y. B. Thakkar

Department of Mathematics,
Hemchandracharya North Gujarat University,
Patan - 384265, Gujarat, INDIA

E-mail : chirag.barasara@gmail.com, yogesh.b.thakkar21@gmail.com

(**Received:** *Apr. 08, 2022* **Accepted:** *Aug. 23, 2022* **Published:** *Aug. 30, 2022)*

Special Issue
Proceedings of National Conference on
"Emerging Trends in Discrete Mathematics, NCETDM - 2022"

Abstract: In 1987, Cahit [7] introduced cordial labeling as a weaker version of graceful labeling and harmonious labeling. In 2011, Varatharajan *et al.* [16] have introduced divisor cordial labeling as a variant of cordial labeling. In this paper, we investigate divisor cordial labeling for C_n-quadrilateral snake, degree splitting graph of triangular snake, quadrilateral snake, ladder graph and helm graph. Moreover, we discuss divisor cordial labeling of graphs obtained by duplication of an arbitrary edge by a new vertex in path, duplication of an arbitrary vertex by a new edge in path, duplication of an arbitrary vertex by a new vertex in path and duplication of an arbitrary edge by a new edge in path.

Keywords and Phrases: Graph Labeling, Cordial Labeling, Divisor Cordial Labeling, Graph Operation, Degree Splitting Graph, Duplication of Graph Elements, Snake Graph.

2020 Mathematics Subject Classification: 05C78, 05C76, 05C38.

1. Introduction and Preliminaries

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$. For all standard terminologies and notations we follow Clark and Holton [8]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1. *A labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a vertex labeling (an edge labeling).*
For an extensive survey on graph labeling and bibliographic references we refer to Gallian [9].
The variants of cordial labeling are commonly known as equitable labeling. One of the equitable labeling called divisor cordial labeling was defined by Varatharajan *et al.* [16] as follows:

Definition 1.2. *For a graph $G = (V(G), E(G))$, the vertex labeling function is defined as a bijection $f : V(G) \to \{1, 2, \dots, |V(G)|\}$ such that induced edge labeling function $f^* : E(G) \to \{0, 1\}$ is given by*

$$f^*(e = uv) = \begin{cases} 1, & \text{if } f(u)/f(v) \text{ or } f(v)/f(u); \\ 0, & \text{otherwise.} \end{cases}$$

Denote the number of edges labeled with 0 and 1 by $e_f(0)$ and $e_f(1)$ respectively. f is called divisor cordial labeling of graph G if $|e_f(0) - e_f(1)| \leq 1$. The graph that admits a divisor cordial labeling is called a divisor cordial graph.
Varatharajan *et al.* [16, 17] have investigated divisor cordial labeling for path, cycle, wheel, star and some complete bipartite graphs. Vaidya and Shah [14, 15] have proved many results related to divisor cordial labeling for the larger graphs obtained using graph operations like splitting, degree splitting, shadow and square graph of graph.
Bosamia and Kanani [5, 6] have discussed divisor cordial labeling in the context of corona product and some operations on bistar.
Murugan and Devakiruba [10] as well as Rokad and Ghodasara [12] have obtained divisor cordial labeling for some cycle related graphs. Divisor cordial labeling for duplication of graph elements is studied by Thirusangu and Madhu [13]. Prajapati and Patel [11] have discussed divisor cordial labeling in context of friendship graph. Barasara and Thakkar [1, 2] have derived results related to divisor cordial labeling for some cycle and wheel related graphs. Also, they have discussed divisor cordial labeling of graphs in the context of some graph operations. In [3] they have studied divisor cordial labeling for ladders and total graph of some graphs.
In this paper, we investigate divisor cordial labeling for C_n-quadrilateral snake as well as larger graphs obtain using graph operations.

2. Divisor Cordial Labeling of Snake Graph

Definition 2.1. *The C_n-triangular snake is obtained from the cycle C_n by replacing every edge of a cycle by a cycle C_3.*

Definition 2.2. *The C_n-quadrilateral snake is obtained from the cycle C_n by replacing every edge of a cycle by a cycle C_4.*

Theorem 2.3. *The C_n-quadrilateral snake is a divisor cordial graph.*
Proof. Let C_n be the cycle with vertices $v_1, v_4, v_7, ..., v_{3n-2}$. Now to construct C_n-quadrilateral snake graph G, join v_{3i-2} to a new vertex v_{3i-1}, v_{3i+1} to a new vertex v_{3i} and v_{3i-1} to v_{3i} for $i = 1, 2, ..., n$. Then $|V(G)| = 3n$ and $|E(G)| = 4n$. We define the divisor cordial labeling $f : V(G) \to \{1, 2, \ldots, 3n\}$ as follows.

$f(v_{3n-2}) = 1$,
$f(v_i) = 1 \times 1 \times 2^{p_0+1-i}$; for $1 \le i \le p_0$,
where p_0 is greatest integer such that $2^{p_0} \le 3n$,
$f(v_{i+p_0}) = 3 \times 1 \times 2^{p_1+1-i}$; for $1 \le i \le p_1 + 1$,
where p_1 is greatest integer such that $3 \times 2^{p_1} \le 3n$,
$f(v_{i+p_0+p_1+1}) = 3 \times 3 \times 2^{p_2+1-i}$; for $1 \le i \le p_2 + 1$,
where p_2 is greatest integer such that $3 \times 3 \times 2^{p_2} \le 3n$,
$f(v_{i+p_0+p_1+p_2+2}) = 3 \times 3^2 \times 2^{p_3+1-i}$; for $1 \le i \le p_3 + 1$,
where p_3 is greatest integer such that $3 \times 3^2 \times 2^{p_3} \le 3n$,
$f(v_{i+p_0+p_1+p_2+p_3+3}) = 3 \times 3^3 \times 2^{p_4+1-i}$; for $1 \le i \le p_4 + 1$,
where p_4 is greatest integer such that $3 \times 3^4 \times 2^{p_4} \le 3n$,

Continuing for m_1 terms such that m_1 is greatest integer satisfying $3^{m_1} \le 3n$ or we get at least $2n - 1$ edges with label 1,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1}) = 5 \times 1 \times 2^{q_0+1-i}$; for $1 \le i \le q_0 + 1$,
where q_0 is greatest integer such that $5 \times 1 \times 2^{q_0} \le 3n$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+1}) = 5 \times 3 \times 2^{q_1+1-i}$; for $1 \le i \le q_1 + 1$,
where q_1 is greatest integer such that $5 \times 3 \times 2^{q_1} \le 3n$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+2}) = 5 \times 3^2 \times 2^{q_2+1-i}$; for $1 \le i \le q_2 + 1$,
where q_2 is greatest integer such that $5 \times 3^2 \times 2^{q_2} \le 3n$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+q_2+3}) = 5 \times 3^3 \times 2^{q_3+1-i}$; for $1 \le i \le q_3 + 1$,
where q_3 is greatest integer such that $5 \times 3^3 \times 2^{q_3} \le 3n$,

Continuing for m_2 terms such that m_2 is greatest integer satisfying $5 \times 3^{m_2} \le 3n$ or we get at least $2n - 1$ edges with label 1,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+...+q_{m_2}+m_2+1}) = 7 \times 1 \times 2^{r_0+1-i}$; for $1 \le i \le r_0 + 1$,
where r_0 is greatest integer such that $7 \times 1 \times 2^{r_0} \le 3n$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+...+q_{m_2}+m_2+1+r_0+1}) = 7 \times 3 \times 2^{r_1+1-i}$; for $1 \le i \le r_1 + 1$,
where r_1 is greatest integer such that $7 \times 3 \times 2^{r_1} \le 3n$.

Continuing in this way till we get at least $2n-1$ edges with label 1.
If we get $2n-1$ edges with label 1, take $f(v_{3n-1}) = 2p'$ and $f(v_{3n}) = p'$, where p' is largest prime number such that $2p' \leq 3n$.
Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).
In view of above defined labeling pattern, we have $e_f(0) = e_f(1) = 2n$. Thus, $|e_f(0) - e_f(1)| = 0$.
Hence, the C_n-quadrilateral snake is a divisor cordial graph.

Illustration 2.4. *The C_4-quadrilateral snake and its divisor cordial labeling is shown in Fig 1.*

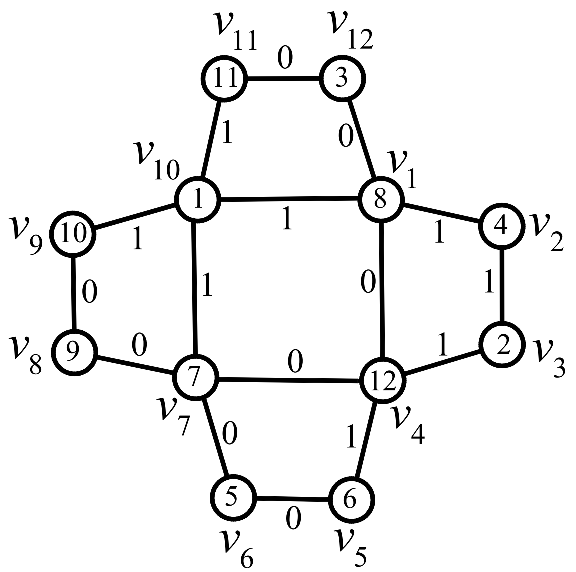

Fig 1: The C_4-quadrilateral snake and its divisor cordial labeling.

3. Divisor Cordial Labeling for Some Degree Splitting Graph

Definition 3.1. *Let $G = (V(G), E(G))$ be a graph with $V = S_1 \cup S_2 \cup ...S_t \cup T$ where each S_i is a set of all vertices having same degree (at least two vertices) and $T = V \setminus \cup S_i$. The degree splitting graph of G denoted by $DS(G)$ is obtained from the graph G by adding vertices $w_1, w_2, ..., w_t$ and joining to each vertex of S_i for $1 \leq i \leq t$.*

Definition 3.2. *The triangular snake T_n is obtained from the path P_n by replacing every edge of a path by a cycle C_3.*

Theorem 3.3. *The graph $DS(T_n)$ is a divisor cordial graph for $n \geq 4$.*

Proof. Let P_n be the path with vertices $v_1, v_3, v_5, ..., v_{2n-1}$. To construct T_n from P_n, join vertices v_{2i-1} and v_{2i+1} to a new vertex v_{2i} for $i = 1, 2, ..., n-1$. Now to obtain $DS(T_n)$, join v_1, v_{2n-1} and v_{2i} to a new vertex v' for $i = 1, 2, ..., n-1$ and join v_{2i-1} to a new vertex v'' for $i = 2, 3, ..., n-2$. Then $|V(DS(T_n))| = 2n+1$ and $|E(DS(T_n))| = 5n-4$.

We define the divisor cordial labeling $f : V(DS(T_n)) \to \{1, 2, \ldots, 2n+1\}$ as follows.

$f(v') = 1$,
$f(v_i) = 1 \times 1 \times 2^{p_0+1-i}$; for $1 \le i \le p_0$,
where p_0 is greatest integer such that $2^{p_0} \le 2n+1$,
$f(v_{i+p_0}) = 3 \times 1 \times 2^{p_1+1-i}$; for $1 \le i \le p_1+1$,
where p_1 is greatest integer such that $3 \times 2^{p_1} \le 2n+1$,
$f(v_{i+p_0+p_1+1}) = 3 \times 3 \times 2^{p_2+1-i}$; for $1 \le i \le p_2+1$,
where p_2 is greatest integer such that $3 \times 3 \times 2^{p_2} \le 2n+1$,
$f(v_{i+p_0+p_1+p_2+2}) = 3 \times 3^2 \times 2^{p_3+1-i}$; for $1 \le i \le p_3+1$,
where p_3 is greatest integer such that $3 \times 3^2 \times 2^{p_3} \le 2n+1$,
$f(v_{i+p_0+p_1+p_2+p_3+3}) = 3 \times 3^3 \times 2^{p_4+1-i}$; for $1 \le i \le p_4+1$,
where p_4 is greatest integer such that $3 \times 3^4 \times 2^{p_4} \le 2n+1$,

Continuing for m_1 terms such that m_1 is greatest integer satisfying $3^{m_1} \le 2n+1$ or we get at least $\left\lfloor \frac{5n-4}{2} \right\rfloor - 1$ edges with label 1,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1}) = 5 \times 1 \times 2^{q_0+1-i}$; for $1 \le i \le q_0+1$,
where q_0 is greatest integer such that $5 \times 1 \times 2^{q_0} \le 2n+1$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+1}) = 5 \times 3 \times 2^{q_1+1-i}$; for $1 \le i \le q_1+1$,
where q_1 is greatest integer such that $5 \times 3 \times 2^{q_1} \le 2n+1$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+2}) = 5 \times 3^2 \times 2^{q_2+1-i}$; for $1 \le i \le q_2+1$,
where q_2 is greatest integer such that $5 \times 3^2 \times 2^{q_2} \le 2n+1$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+q_2+3}) = 5 \times 3^3 \times 2^{q_3+1-i}$; for $1 \le i \le q_3+1$,
where q_3 is greatest integer such that $5 \times 3^3 \times 2^{q_3} \le 2n+1$,

Continuing for m_2 terms such that m_2 is greatest integer satisfying $5 \times 3^{m_2} \le 2n+1$ or we get at least $\left\lfloor \frac{5n-4}{2} \right\rfloor - 1$ edges with label 1,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+...+q_{m_2}+m_2+1}) = 7 \times 1 \times 2^{r_0+1-i}$; for $1 \le i \le r_0+1$,
where r_0 is greatest integer such that $7 \times 1 \times 2^{r_0} \le 2n+1$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+...+q_{m_2}+m_2+1+r_0+1}) = 7 \times 3 \times 2^{r_1+1-i}$; for $1 \le i \le r_1+1$,
where r_1 is greatest integer such that $7 \times 3 \times 2^{r_1} \le 2n+1$.

Continuing in this way till we get at least $\left\lfloor \frac{5n-4}{2} \right\rfloor - 1$ edges with label 1.

If we get $\left\lfloor \dfrac{5n-4}{2} \right\rfloor - 1$ edges with label 1, take $f(v_{2n-1}) = 2p'$ and $f(v_{2n-2}) = p'$, where p' is largest prime number such that $2p' < 2n+1$.
Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).
In view of above defined labeling pattern, we have $e_f(0) = \left\lceil \dfrac{5n-4}{2} \right\rceil$ and $e_f(1) = \left\lfloor \dfrac{5n-4}{2} \right\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1$.
Hence, the graph $DS(T_n)$ is a divisor cordial graph for $n \geq 4$.

Illustration 3.4. *The graph $DS(T_6)$ and its divisor cordial labeling is shown in Fig 2.*

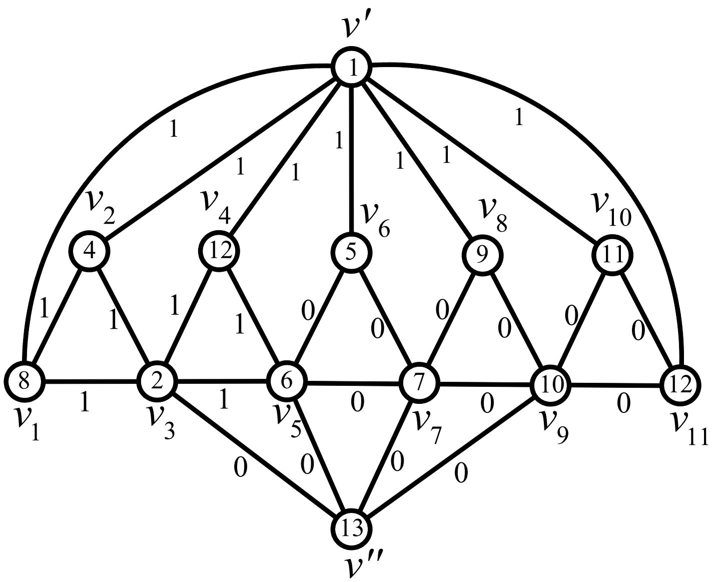

Fig 2: The graph $DS(T_6)$ and its divisor cordial labeling.

Definition 3.5. *The quadrilateral snake Q_n is obtained from the path P_n by replacing every edge of a path by a cycle C_4.*

Theorem 3.6. *The graph $DS(Q_n)$ is a divisor cordial graph for $n \geq 4$.*
Proof. Let P_n be the path with vertices $v_1, v_4, v_7, ..., v_{3n-2}$. To construct Q_n from P_n, join v_{3i-2} to a new vertex v_{3i-1}, v_{3i+1} to a new vertex v_{3i} and v_{3i-1} to v_{3i} for $i = 1, 2, ..., n-1$. Now to obtain $DS(Q_n)$, join v_1, v_{3i-1}, v_{3i} and v_{3n-2} to a new vertex v' for $i = 1, 2, ..., n-1$ and v_{3i-2} to a new vertex v'' $i = 2, 3, ..., n-2$. Then $|V(DS(Q_n))| = 3n$ and $|E(DS(Q_n))| = 7n - 6$.

We define the divisor cordial labeling $f : V(DS(Q_n)) \rightarrow \{1, 2, \ldots, 3n\}$ as follows.

$f(v') = 1,$

$f(v_i) = 1 \times 1 \times 2^{p_0+1-i};$ for $1 \leq i \leq p_0,$

where p_0 is greatest integer such that $2^{p_0} \leq 3n,$

$f(v_{i+p_0}) = 3 \times 1 \times 2^{p_1+1-i};$ for $1 \leq i \leq p_1 + 1,$

where p_1 is greatest integer such that $3 \times 2^{p_1} \leq 3n,$

$f(v_{i+p_0+p_1+1}) = 3 \times 3 \times 2^{p_2+1-i};$ for $1 \leq i \leq p_2 + 1,$

where p_2 is greatest integer such that $3 \times 3 \times 2^{p_2} \leq 3n,$

$f(v_{i+p_0+p_1+p_2+2}) = 3 \times 3^2 \times 2^{p_3+1-i};$ for $1 \leq i \leq p_3 + 1,$

where p_3 is greatest integer such that $3 \times 3^2 \times 2^{p_3} \leq 3n,$

$f(v_{i+p_0+p_1+p_2+p_3+3}) = 3 \times 3^3 \times 2^{p_4+1-i};$ for $1 \leq i \leq p_4 + 1,$

where p_4 is greatest integer such that $3 \times 3^4 \times 2^{p_4} \leq 3n,$

Continuing for m_1 terms such that m_1 is greatest integer satisfying $3^{m_1} \leq 3n$ or we get at least $\left\lfloor \dfrac{7n-6}{2} \right\rfloor - 1$ edges with label 1,

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1}) = 5 \times 1 \times 2^{q_0+1-i};$ for $1 \leq i \leq q_0 + 1,$

where q_0 is greatest integer such that $5 \times 1 \times 2^{q_0} \leq 3n,$

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+1}) = 5 \times 3 \times 2^{q_1+1-i};$ for $1 \leq i \leq q_1 + 1,$

where q_1 is greatest integer such that $5 \times 3 \times 2^{q_1} \leq 3n,$

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+2}) = 5 \times 3^2 \times 2^{q_2+1-i};$ for $1 \leq i \leq q_2 + 1,$

where q_2 is greatest integer such that $5 \times 3^2 \times 2^{q_2} \leq 3n,$

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+q_2+3}) = 5 \times 3^3 \times 2^{q_3+1-i};$ for $1 \leq i \leq q_3 + 1,$

where q_3 is greatest integer such that $5 \times 3^3 \times 2^{q_3} \leq 3n,$

Continuing for m_2 terms such that m_2 is greatest integer satisfying $5 \times 3^{m_2} \leq 3n$ or we get at least $\left\lfloor \dfrac{7n-6}{2} \right\rfloor - 1$ edges with label 1,

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1}) = 7 \times 1 \times 2^{r_0+1-i};$ for $1 \leq i \leq r_0 + 1,$

where r_0 is greatest integer such that $7 \times 1 \times 2^{r_0} \leq 3n,$

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1+r_0+1}) = 7 \times 3 \times 2^{r_1+1-i};$ for $1 \leq i \leq r_1 + 1,$

where r_1 is greatest integer such that $7 \times 3 \times 2^{r_1} \leq 3n.$

Continuing in this way till we get at least $\left\lfloor \dfrac{7n-6}{2} \right\rfloor - 1$ edges with label 1.

If we get $\left\lfloor \dfrac{7n-6}{2} \right\rfloor - 1$ edges with label 1, take $f(v_{3n-2}) = 2p'$ and $f(v_{3n-3}) = p'$, where p' is largest prime number such that $2p' < 3n$.

Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).

In view of above defined labeling pattern, we have $e_f(0) = \left\lceil \frac{7n-6}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{7n-6}{2} \right\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the graph $DS(Q_n)$ is a divisor cordial graph for $n \geq 4$.

Illustration 3.7. *The graph $DS(Q_5)$ and its divisor cordial labeling is shown in Fig 3.*

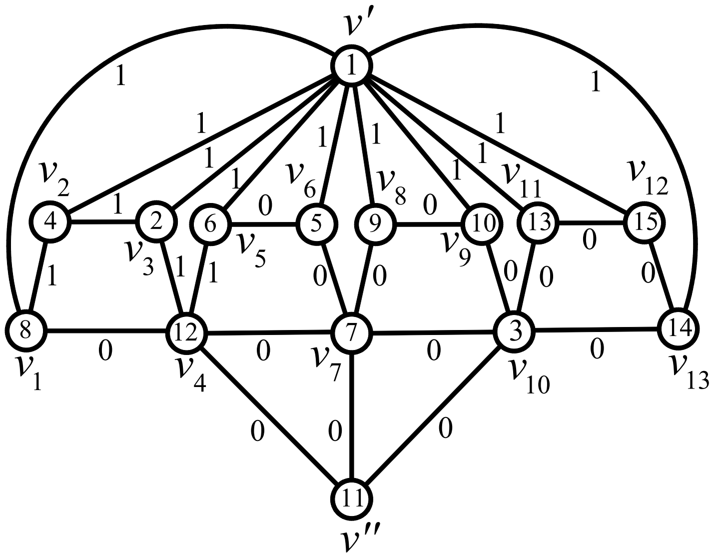

Fig 3: The graph $DS(Q_5)$ and its divisor cordial labeling.

Definition 3.8. *Let G and H be two graphs. The cartesian product of G and H, denoted by $G\Box H$, has the vertex set $V(G) \times V(H)$ and (g, h) is adjacent to (g', h') if $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$.*

Definition 3.9. *The ladder graph L_n is defined as $P_2\Box P_n$.*

Theorem 3.10. *The graph $DS(L_n)$ is a divisor cordial graph for $n \geq 3$.*

Proof. Let v_1, v_4, v_5, ..., v_{4k}, v_{4k+1}, ... be the vertices of first path and v_2, v_3, v_6, ..., v_{4k+2}, v_{4k+3}, ... be the vertices of second path in ladder L_n. Now to obtain $DS(L_n)$, join v_1, v_2, v_{2n-1} and v_{2n} to a new vertex v' and all other vertices to a new vertex v''. Then $|V(DS(L_n))| = 2n + 2$ and $|E(DS(L_n))| = 5n - 2$.

We define the divisor cordial labeling $f : V(DS(L_n)) \to \{1, 2, \ldots, 2n+2\}$ as follows.

$f(v'') = 1,$
$f(v_i) = 1 \times 1 \times 2^{p_0+1-i};$ for $1 \leq i \leq p_0,$
where p_0 is greatest integer such that $2^{p_0} \leq 2n+2,$
$f(v_{i+p_0}) = 3 \times 1 \times 2^{p_1+1-i};$ for $1 \leq i \leq p_1+1,$
where p_1 is greatest integer such that $3 \times 2^{p_1} \leq 2n+2,$
$f(v_{i+p_0+p_1+1}) = 3 \times 3 \times 2^{p_2+1-i};$ for $1 \leq i \leq p_2+1,$
where p_2 is greatest integer such that $3 \times 3 \times 2^{p_2} \leq 2n+2,$
$f(v_{i+p_0+p_1+p_2+2}) = 3 \times 3^2 \times 2^{p_3+1-i};$ for $1 \leq i \leq p_3+1,$
where p_3 is greatest integer such that $3 \times 3^2 \times 2^{p_3} \leq 2n+2,$
$f(v_{i+p_0+p_1+p_2+p_3+3}) = 3 \times 3^3 \times 2^{p_4+1-i};$ for $1 \leq i \leq p_4+1,$
where p_4 is greatest integer such that $3 \times 3^4 \times 2^{p_4} \leq 2n+2,$
Continuing for m_1 terms such that m_1 is greatest integer satisfying $3^{m_1} \leq 2n+2$ or we get at least $\left\lfloor \frac{5n-2}{2} \right\rfloor - 1$ edges with label 1,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1}) = 5 \times 1 \times 2^{q_0+1-i};$ for $1 \leq i \leq q_0+1,$
where q_0 is greatest integer such that $5 \times 1 \times 2^{q_0} \leq 2n+2,$
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+1}) = 5 \times 3 \times 2^{q_1+1-i};$ for $1 \leq i \leq q_1+1,$
where q_1 is greatest integer such that $5 \times 3 \times 2^{q_1} \leq 2n+2,$
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+2}) = 5 \times 3^2 \times 2^{q_2+1-i};$ for $1 \leq i \leq q_2+1,$
where q_2 is greatest integer such that $5 \times 3^2 \times 2^{q_2} \leq 2n+2,$
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+q_2+3}) = 5 \times 3^3 \times 2^{q_3+1-i};$ for $1 \leq i \leq q_3+1,$
where q_3 is greatest integer such that $5 \times 3^3 \times 2^{q_3} \leq 2n+2,$
Continuing for m_2 terms such that m_2 is greatest integer satisfying $5 \times 3^{m_2} \leq 2n+2$ or we get at least $\left\lfloor \frac{5n-2}{2} \right\rfloor - 1$ edges with label 1,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1}) = 7 \times 1 \times 2^{r_0+1-i};$ for $1 \leq i \leq r_0+1,$
where r_0 is greatest integer such that $7 \times 1 \times 2^{r_0} \leq 2n+2,$
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1+r_0+1}) = 7 \times 3 \times 2^{r_1+1-i};$ for $1 \leq i \leq r_1+1,$
where r_1 is greatest integer such that $7 \times 3 \times 2^{r_1} \leq 2n+2.$

Continuing in this way till we get at least $\left\lfloor \frac{5n-2}{2} \right\rfloor - 1$ edges with label 1.

If we get $\left\lfloor \frac{5n-2}{2} \right\rfloor - 1$ edges with label 1, take $f(v_{2n-1}) = 2p'$ and $f(v_{2n}) = p'$, where p' is largest prime number such that $2p' < 2n+2$.
Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).

In view of above defined labeling pattern, we have $e_f(0) = \left\lceil \frac{5n-2}{2} \right\rceil$ and $e_f(1) =$

$\left\lfloor \dfrac{5n-2}{2} \right\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1$.
Hence, the graph $DS(L_n)$ is a divisor cordial graph for $n \geq 3$.

Illustration 3.11. *The graph $DS(L_6)$ and its divisor cordial labeling is shown in Fig 4.*

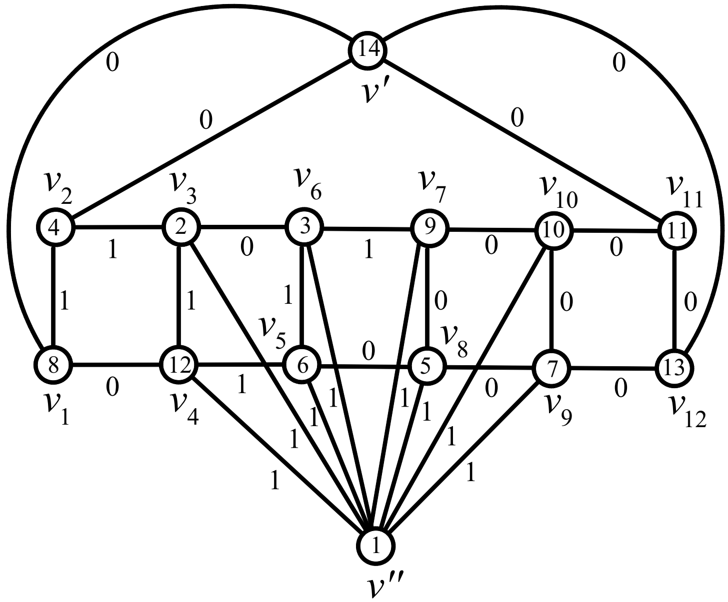

Fig 4: The graph $DS(L_6)$ and its divisor cordial labeling.

Definition 3.12. *The helm H_n is the graph obtained from a wheel by attaching a pendent edge to each of its rim vertices.*

Theorem 3.13. *The graph $DS(H_n)$ is a divisor cordial graph.*
Proof. Let H_n be the helm graph with rim vertices $v_1, v_2, v_3, ..., v_n$, pendent vertices v_i' attached to v_i for $i = 1, 2, ..., n$ and an apex vertex v.
We consider following two cases.
Case 1: For $n = 4$
Case 2: For $n \neq 4$
For the graph $DS(H_n)$, join vertices v_i to a new vertex w_1 and vertices v_i' to a new vertex w_2 for $i = 1, 2, ..., n$. Then $|V(DS(H_n))| = 2n + 3$ and $|E(DS(H_n))| = 5n$. We define the divisor cordial labeling $f : V(DS(H_n)) \rightarrow \{1, 2, \ldots, 2n + 3\}$ by following two subcases.
Subcase 1: For even n

$f(w_1) = 1,$
$f(v) = 2,$
$f(v_i) = 2i + 2; \quad \text{for } 1 \leq i \leq n,$
$f(v'_{2i}) = 2i + 1; \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.$

Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).

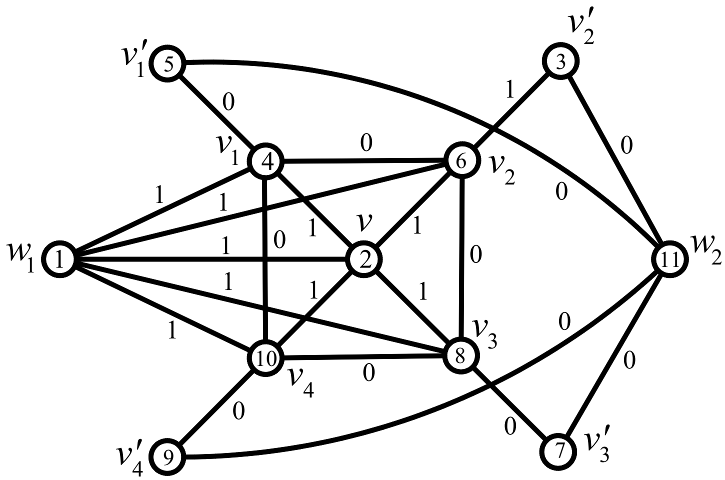

Fig 5: The graph $DS(H_4)$ and its divisor cordial labeling.

Subcase 2: For odd n

$f(w_1) = 1,$
$f(v) = 2,$
$f(v_i) = 2i + 2; \quad \text{for } 1 \leq i \leq n,$
$f(v'_{2i}) = 2i + 1; \quad \text{for } 1 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor.$

Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).

In each subcase, we have $e_f(0) = \left\lceil \frac{5n}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{5n}{2} \right\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the graph $DS(H_n)$ is a divisor cordial graph.

Illustration 3.14. *The graph $DS(H_7)$ and its divisor cordial labeling is shown in Fig 6.*

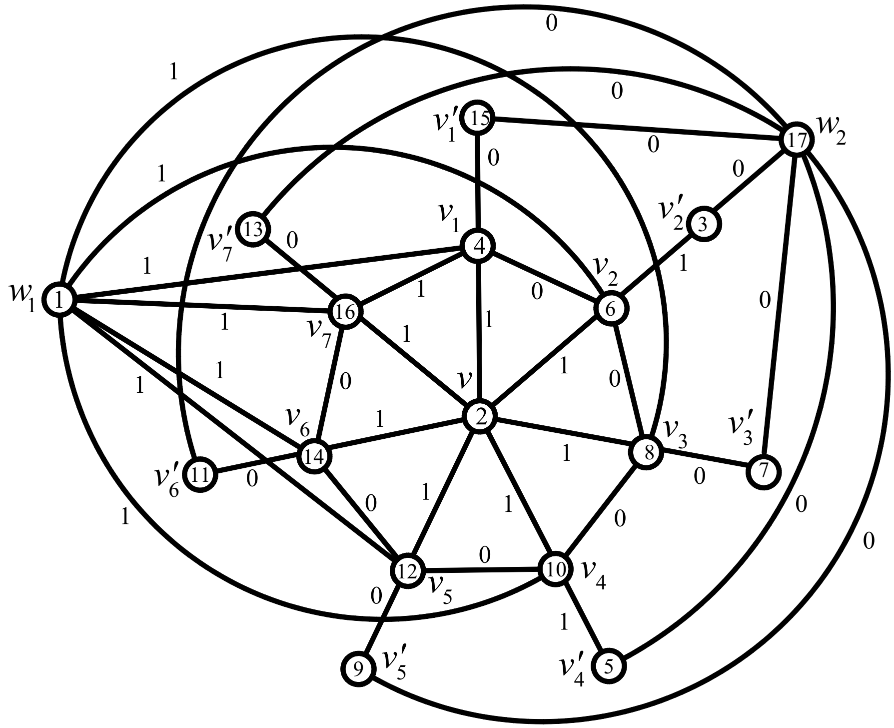

Fig 6: The graph $DS(H_7)$ and its divisor cordial labeling.

4. Divisor Cordial Labeling in the Context of Duplication Operation

Definition 4.1. *Two adjacent vertices are called neighbours. The set of all neighbours of vertex v is called the neighbourhood set of v. It is denoted by $N(v)$.*

Definition 4.2. *For a graph G, duplication of an edge $e = uv$ by a new vertex v' produces a new graph G'such that $N(v') = \{u, v\}$.*

Theorem 4.3. *The graph obtained by duplication of an arbitrary edge by a new vertex in path P_n is a divisor cordial graph.*

Proof. Let P_n be the path with vertices $v_1, v_2, v_3, ..., v_n$. Let the graph G be obtained by duplication of an arbitrary edge e_k in P_n by a new vertex v. Then $|V(G)| = n + 1$ and $|E(G)| = n + 1$.

We define the divisor cordial labeling $f : V(G) \rightarrow \{1, 2, \ldots, n + 1\}$ by following two cases.

Case 1: For $n = 2$.

The graph G is isomorphic to cycle C_3 and Varatharajan *et al.* [16] proved that cycles are divisor cordial graph. Hence, G is divisor cordial graph.

Case 2: For $n \geq 3$.

$f(v) = 1$,
$f(v_i) = 1 \times 1 \times 2^{p_0+1-i}$; for $1 \leq i \leq p_0$,
where p_0 is greatest integer such that $2^{p_0} \leq n+1$,
$f(v_{i+p_0}) = 3 \times 1 \times 2^{p_1+1-i}$; for $1 \leq i \leq p_1+1$,
where p_1 is greatest integer such that $3 \times 2^{p_1} \leq n+1$,
$f(v_{i+p_0+p_1+1}) = 3 \times 3 \times 2^{p_2+1-i}$; for $1 \leq i \leq p_2+1$,
where p_2 is greatest integer such that $3 \times 3 \times 2^{p_2} \leq n+1$,
$f(v_{i+p_0+p_1+p_2+2}) = 3 \times 3^2 \times 2^{p_3+1-i}$; for $1 \leq i \leq p_3+1$,
where p_3 is greatest integer such that $3 \times 3^2 \times 2^{p_3} \leq n+1$,
$f(v_{i+p_0+p_1+p_2+p_3+3}) = 3 \times 3^3 \times 2^{p_4+1-i}$; for $1 \leq i \leq p_4+1$,
where p_4 is greatest integer such that $3 \times 3^4 \times 2^{p_4} \leq n+1$,

Continuing for m_1 terms such that m_1 is greatest integer satisfying $3^{m_1} \leq n+1$ or we get $\left\lfloor \frac{n+1}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1}) = 5 \times 1 \times 2^{q_0+1-i}$; for $1 \leq i \leq q_0+1$,
where q_0 is greatest integer such that $5 \times 1 \times 2^{q_0} \leq n+1$,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+1}) = 5 \times 3 \times 2^{q_1+1-i}$; for $1 \leq i \leq q_1+1$,
where q_1 is greatest integer such that $5 \times 3 \times 2^{q_1} \leq n+1$,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+2}) = 5 \times 3^2 \times 2^{q_2+1-i}$; for $1 \leq i \leq q_2+1$,
where q_2 is greatest integer such that $5 \times 3^2 \times 2^{q_2} \leq n+1$,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+q_2+3}) = 5 \times 3^3 \times 2^{q_3+1-i}$; for $1 \leq i \leq q_3+1$,
where q_3 is greatest integer such that $5 \times 3^3 \times 2^{q_3} \leq n+1$,

Continuing for m_2 terms such that m_2 is greatest integer satisfying $5 \times 3^{m_2} \leq n+1$ or we get $\left\lfloor \frac{n+1}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1}) = 7 \times 1 \times 2^{r_0+1-i}$; for $1 \leq i \leq r_0+1$,
where r_0 is greatest integer such that $7 \times 1 \times 2^{r_0} \leq n+1$,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1+r_0+1}) = 7 \times 3 \times 2^{r_1+1-i}$; for $1 \leq i \leq r_1+1$,
where r_1 is greatest integer such that $7 \times 3 \times 2^{r_1} \leq n+1$.

Continuing in this way till we get $\left\lfloor \frac{n+1}{2} \right\rfloor$ edges with label 1.

Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).

In view of above defined labeling pattern, we have $e_f(0) = \left\lceil \frac{n+1}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{n+1}{2} \right\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the graph obtained by duplication of an arbitrary edge by a new vertex in path P_n is a divisor cordial graph.

Illustration 4.4. *The graph obtained by duplication of an edge e_2 by a new vertex in path P_7 and its divisor cordial labeling is shown in Fig 7.*

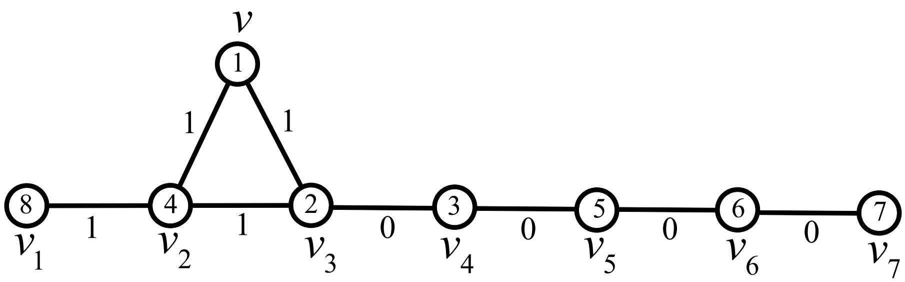

Fig 7: The graph obtained by duplication of an edge e_2 by a new vertex in path P_7 and its divisor cordial labeling.

Definition 4.5. *For a graph G, duplication of a vertex v by a new edge $e' = v'v''$ produces a new graph G' such that $N(v') = \{v, v''\}$ and $N(v'') = \{v, v'\}$.*

Theorem 4.6. *The graph obtained by duplication of an arbitrary vertex by a new edge in path P_n is a divisor cordial graph.*

Proof. Let P_n be the path with vertices $v_1, v_2, v_3, ..., v_n$. Let the graph G be obtained by duplication of an arbitrary vertex v_k in P_n by a new edge $e = v_1'v_2'$. Then $|V(G)| = n + 2$ and $|E(G)| = n + 2$.

We define the divisor cordial labeling $f : V(G) \to \{1, 2, \ldots, n + 2\}$ as follows.

$f(v_1') = 1,$

$f(v_i) = 1 \times 1 \times 2^{p_0+1-i};$ for $1 \le i \le p_0,$

where p_0 is greatest integer such that $2^{p_0} \le n + 2,$

$f(v_{i+p_0}) = 3 \times 1 \times 2^{p_1+1-i};$ for $1 \le i \le p_1 + 1,$

where p_1 is greatest integer such that $3 \times 2^{p_1} \le n + 2,$

$f(v_{i+p_0+p_1+1}) = 3 \times 3 \times 2^{p_2+1-i};$ for $1 \le i \le p_2 + 1,$

where p_2 is greatest integer such that $3 \times 3 \times 2^{p_2} \le n + 2,$

$f(v_{i+p_0+p_1+p_2+2}) = 3 \times 3^2 \times 2^{p_3+1-i};$ for $1 \le i \le p_3 + 1,$

where p_3 is greatest integer such that $3 \times 3^2 \times 2^{p_3} \le n + 2,$

$f(v_{i+p_0+p_1+p_2+p_3+3}) = 3 \times 3^3 \times 2^{p_4+1-i};$ for $1 \le i \le p_4 + 1,$

where p_4 is greatest integer such that $3 \times 3^4 \times 2^{p_4} \le n + 2,$

Continuing for m_1 terms such that m_1 is greatest integer satisfying $3^{m_1} \le n+2$ or we get $\left\lfloor \frac{n+2}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1}) = 5 \times 1 \times 2^{q_0+1-i}$; for $1 \le i \le q_0+1$,
where q_0 is greatest integer such that $5 \times 1 \times 2^{q_0} \le n+2$,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+1}) = 5 \times 3 \times 2^{q_1+1-i}$; for $1 \le i \le q_1+1$,
where q_1 is greatest integer such that $5 \times 3 \times 2^{q_1} \le n+2$,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+2}) = 5 \times 3^2 \times 2^{q_2+1-i}$; for $1 \le i \le q_2+1$,
where q_2 is greatest integer such that $5 \times 3^2 \times 2^{q_2} \le n+2$,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+q_2+3}) = 5 \times 3^3 \times 2^{q_3+1-i}$; for $1 \le i \le q_3+1$,
where q_3 is greatest integer such that $5 \times 3^3 \times 2^{q_3} \le n+2$,
Continuing for m_2 terms such that m_2 is greatest integer satisfying $5 \times 3^{m_2} \le n+2$ or we get $\left\lfloor \frac{n+2}{2} \right\rfloor$ edges with label 1,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1}) = 7 \times 1 \times 2^{r_0+1-i}$; for $1 \le i \le r_0+1$,
where r_0 is greatest integer such that $7 \times 1 \times 2^{r_0} \le n+2$,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1+r_0+1}) = 7 \times 3 \times 2^{r_1+1-i}$; for $1 \le i \le r_1+1$,
where r_1 is greatest integer such that $7 \times 3 \times 2^{r_1} \le n+2$.

Continuing in this way till we get $\left\lfloor \frac{n+2}{2} \right\rfloor$ edges with label 1.

Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).

In view of above defined labeling pattern, we have $e_f(0) = \left\lceil \frac{n+2}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{n+2}{2} \right\rfloor$. Thus, $|e_f(0) - e_f(1)| \le 1$.

Hence, the graph obtained by duplication of an arbitrary vertex by a new edge in path P_n is a divisor cordial graph.

Illustration 4.7. *The graph obtained by duplication of a vertex v_3 by a new edge in path P_8 and its divisor cordial labeling is shown in Fig 8.*

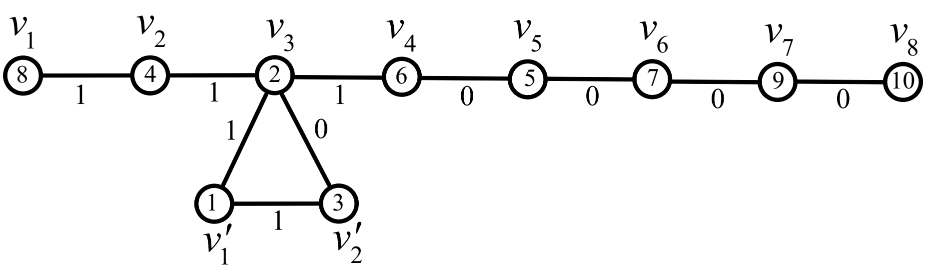

Fig 8: The graph obtained by duplication of a vertex v_3 by a new edge in path P_8 and its divisor cordial labeling.

Definition 4.8. *For a graph G, duplication of a vertex v by a new vertex v' produces a new graph G' such that $N(v) = N(v')$.*

Theorem 4.9. *The graph obtained by duplication of an arbitrary vertex by a new vertex in path P_n is a divisor cordial graph.*

Proof. Let P_n be the path with vertices $v_1, v_2, v_3, ..., v_n$. Let the graph G be obtained by duplication of an arbitrary vertex v by a new vertex v'.

We define the divisor cordial labeling $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ by following two cases.

Case 1: Duplication of either v_1 or v_n.

Then $|V(G)| = n + 1$ and $|E(G)| = n$.

$f(v') = 1$,

$f(v_i) = 1 \times 1 \times 2^{p_0+1-i}$; for $1 \leq i \leq p_0$,

where p_0 is greatest integer such that $2^{p_0} \leq n + 1$,

$f(v_{i+p_0}) = 3 \times 1 \times 2^{p_1+1-i}$; for $1 \leq i \leq p_1 + 1$,

where p_1 is greatest integer such that $3 \times 2^{p_1} \leq n + 1$,

$f(v_{i+p_0+p_1+1}) = 3 \times 3 \times 2^{p_2+1-i}$; for $1 \leq i \leq p_2 + 1$,

where p_2 is greatest integer such that $3 \times 3 \times 2^{p_2} \leq n + 1$,

$f(v_{i+p_0+p_1+p_2+2}) = 3 \times 3^2 \times 2^{p_3+1-i}$; for $1 \leq i \leq p_3 + 1$,

where p_3 is greatest integer such that $3 \times 3^2 \times 2^{p_3} \leq n + 1$,

$f(v_{i+p_0+p_1+p_2+p_3+3}) = 3 \times 3^3 \times 2^{p_4+1-i}$; for $1 \leq i \leq p_4 + 1$,

where p_4 is greatest integer such that $3 \times 3^4 \times 2^{p_4} \leq n + 1$,

Continuing for m_1 terms such that m_1 is greatest integer satisfying $3^{m_1} \leq n + 1$ or we get $\left\lfloor \frac{n}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1}) = 5 \times 1 \times 2^{q_0+1-i}$; for $1 \leq i \leq q_0 + 1$,

where q_0 is greatest integer such that $5 \times 1 \times 2^{q_0} \leq n + 1$,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+1}) = 5 \times 3 \times 2^{q_1+1-i}$; for $1 \leq i \leq q_1 + 1$,

where q_1 is greatest integer such that $5 \times 3 \times 2^{q_1} \leq n + 1$,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+2}) = 5 \times 3^2 \times 2^{q_2+1-i}$; for $1 \leq i \leq q_2 + 1$,

where q_2 is greatest integer such that $5 \times 3^2 \times 2^{q_2} \leq n + 1$,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+q_2+3}) = 5 \times 3^3 \times 2^{q_3+1-i}$; for $1 \leq i \leq q_3 + 1$,

where q_3 is greatest integer such that $5 \times 3^3 \times 2^{q_3} \leq n + 1$,

Continuing for m_2 terms such that m_2 is greatest integer satisfying $5 \times 3^{m_2} \leq n+1$ or we get $\left\lfloor \frac{n}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+...+q_{m_2}+m_2+1}) = 7 \times 1 \times 2^{r_0+1-i}$; for $1 \leq i \leq r_0 + 1$,

where r_0 is greatest integer such that $7 \times 1 \times 2^{r_0} \leq n + 1$,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+...+q_{m_2}+m_2+1+r_0+1}) = 7 \times 3 \times 2^{r_1+1-i}$; for $1 \leq i \leq r_1 + 1$,

where r_1 is greatest integer such that $7 \times 3 \times 2^{r_1} \leq n + 1$.

Continuing in this way till we get $\left\lfloor \dfrac{n}{2} \right\rfloor$ edges with label 1.

Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).

In view of above defined labeling pattern, we have $e_f(0) = \left\lceil \dfrac{n}{2} \right\rceil$ and $e_f(1) = \left\lfloor \dfrac{n}{2} \right\rfloor$.

Thus, $|e_f(0) - e_f(1)| \leq 1$.

Case 2: Duplication of any vertex other than v_1 and v_n.

Then $|V(G)| = n + 1$ and $|E(G)| = n + 1$.

$f(v') = 1,$

$f(v_i) = 1 \times 1 \times 2^{p_0+1-i};$ for $1 \leq i \leq p_0,$

where p_0 is greatest integer such that $2^{p_0} \leq n + 1,$

$f(v_{i+p_0}) = 3 \times 1 \times 2^{p_1+1-i};$ for $1 \leq i \leq p_1 + 1,$

where p_1 is greatest integer such that $3 \times 2^{p_1} \leq n + 1,$

$f(v_{i+p_0+p_1+1}) = 3 \times 3 \times 2^{p_2+1-i};$ for $1 \leq i \leq p_2 + 1,$

where p_2 is greatest integer such that $3 \times 3 \times 2^{p_2} \leq n + 1,$

$f(v_{i+p_0+p_1+p_2+2}) = 3 \times 3^2 \times 2^{p_3+1-i};$ for $1 \leq i \leq p_3 + 1,$

where p_3 is greatest integer such that $3 \times 3^2 \times 2^{p_3} \leq n + 1,$

$f(v_{i+p_0+p_1+p_2+p_3+3}) = 3 \times 3^3 \times 2^{p_4+1-i};$ for $1 \leq i \leq p_4 + 1,$

where p_4 is greatest integer such that $3 \times 3^4 \times 2^{p_4} \leq n + 1,$

Continuing for m_1 terms such that m_1 is greatest integer satisfying $3^{m_1} \leq n + 1$ or we get $\left\lfloor \dfrac{n+1}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1}) = 5 \times 1 \times 2^{q_0+1-i};$ for $1 \leq i \leq q_0 + 1,$

where q_0 is greatest integer such that $5 \times 1 \times 2^{q_0} \leq n + 1,$

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+1}) = 5 \times 3 \times 2^{q_1+1-i};$ for $1 \leq i \leq q_1 + 1,$

where q_1 is greatest integer such that $5 \times 3 \times 2^{q_1} \leq n + 1,$

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+2}) = 5 \times 3^2 \times 2^{q_2+1-i};$ for $1 \leq i \leq q_2 + 1,$

where q_2 is greatest integer such that $5 \times 3^2 \times 2^{q_2} \leq n + 1,$

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+q_2+3}) = 5 \times 3^3 \times 2^{q_3+1-i};$ for $1 \leq i \leq q_3 + 1,$

where q_3 is greatest integer such that $5 \times 3^3 \times 2^{q_3} \leq n + 1,$

Continuing for m_2 terms such that m_2 is greatest integer satisfying $5 \times 3^{m_2} \leq n+1$ or we get $\left\lfloor \dfrac{n+1}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1}) = 7 \times 1 \times 2^{r_0+1-i};$ for $1 \leq i \leq r_0 + 1,$

where r_0 is greatest integer such that $7 \times 1 \times 2^{r_0} \leq n + 1,$

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1+r_0+1}) = 7 \times 3 \times 2^{r_1+1-i};$ for $1 \leq i \leq r_1 + 1,$

where r_1 is greatest integer such that $7 \times 3 \times 2^{r_1} \leq n + 1.$

Continuing in this way till we get $\left\lfloor \dfrac{n+1}{2} \right\rfloor$ edges with label 1.

Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).

In view of above defined labeling pattern, we have $e_f(0) = \left\lceil \dfrac{n+1}{2} \right\rceil$ and $e_f(1) = \left\lfloor \dfrac{n+1}{2} \right\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the graph obtained by duplication of an arbitrary vertex by a new vertex in path P_n is a divisor cordial graph.

Illustration 4.10. *The graph obtained by duplication of a vertex v_1 by a new vertex in path P_7 and its divisor cordial labeling is shown in Fig 9.*

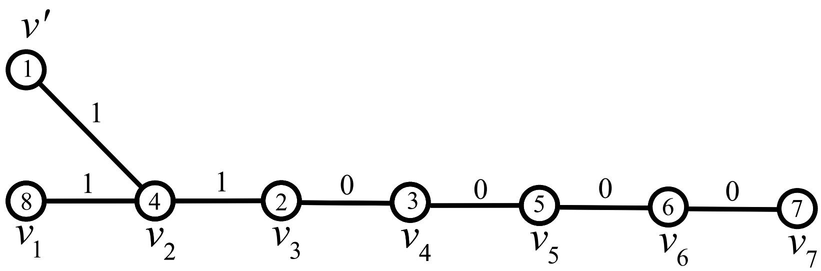

Fig 9: The graph obtained by duplication of a vertex v_1 by a new vertex in path P_7 and its divisor cordial labeling.

Illustration 4.11. *The graph obtained by duplication of a vertex v_4 by a new vertex in path P_8 and its divisor cordial labeling is shown in Fig 10.*

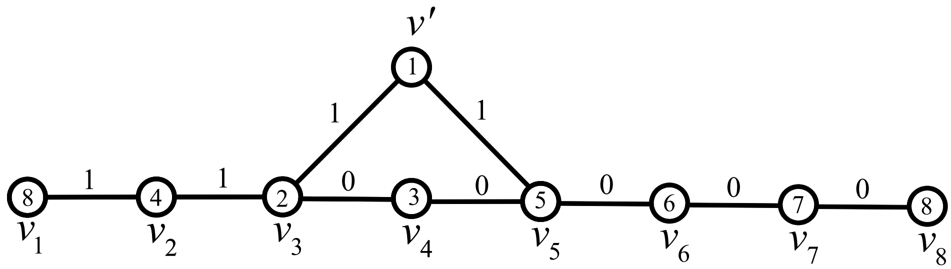

Fig 10: The graph obtained by duplication of a vertex v_4 by a new vertex in path P_8 and its divisor cordial labeling.

Definition 4.12. *For a graph G, duplication of an edge $e = uv$ by a new edge $e' = u'v'$ produces a new graph G' such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$.*

Theorem 4.13. *The graph obtained by duplication of an arbitrary edge by a new edge in path P_n is a divisor cordial graph.*
Proof. Let P_n be the path with vertices $v_1, v_2, v_3, ..., v_n$. Let the graph G be obtained by duplication of an arbitrary edge e_k in P_n by a new edge $e'_k = v'v''$.
We define the divisor cordial labeling $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ by following two cases.

Case 1: Duplication of either e_1 or e_{n-1}.
Then $|V(G)| = n + 2$ and $|E(G)| = n + 1$.

$f(v'') = 1$,
$f(v_i) = 1 \times 1 \times 2^{p_0+1-i}$; for $1 \le i \le p_0$,
where p_0 is greatest integer such that $2^{p_0} \le n + 2$,
$f(v_{i+p_0}) = 3 \times 1 \times 2^{p_1+1-i}$; for $1 \le i \le p_1 + 1$,
where p_1 is greatest integer such that $3 \times 2^{p_1} \le n + 2$,
$f(v_{i+p_0+p_1+1}) = 3 \times 3 \times 2^{p_2+1-i}$; for $1 \le i \le p_2 + 1$,
where p_2 is greatest integer such that $3 \times 3 \times 2^{p_2} \le n + 2$,
$f(v_{i+p_0+p_1+p_2+2}) = 3 \times 3^2 \times 2^{p_3+1-i}$; for $1 \le i \le p_3 + 1$,
where p_3 is greatest integer such that $3 \times 3^2 \times 2^{p_3} \le n + 2$,
$f(v_{i+p_0+p_1+p_2+p_3+3}) = 3 \times 3^3 \times 2^{p_4+1-i}$; for $1 \le i \le p_4 + 1$,
where p_4 is greatest integer such that $3 \times 3^4 \times 2^{p_4} \le n + 2$,

Continuing for m_1 terms such that m_1 is greatest integer satisfying $3^{m_1} \le n + 2$ or we get $\left\lfloor \dfrac{n+1}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1}) = 5 \times 1 \times 2^{q_0+1-i}$; for $1 \le i \le q_0 + 1$,
where q_0 is greatest integer such that $5 \times 1 \times 2^{q_0} \le n + 2$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+1}) = 5 \times 3 \times 2^{q_1+1-i}$; for $1 \le i \le q_1 + 1$,
where q_1 is greatest integer such that $5 \times 3 \times 2^{q_1} \le n + 2$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+2}) = 5 \times 3^2 \times 2^{q_2+1-i}$; for $1 \le i \le q_2 + 1$,
where q_2 is greatest integer such that $5 \times 3^2 \times 2^{q_2} \le n + 2$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+q_2+3}) = 5 \times 3^3 \times 2^{q_3+1-i}$; for $1 \le i \le q_3 + 1$,
where q_3 is greatest integer such that $5 \times 3^3 \times 2^{q_3} \le n + 2$,

Continuing for m_2 terms such that m_2 is greatest integer satisfying $5 \times 3^{m_2} \le n + 2$ or we get $\left\lfloor \dfrac{n+1}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+...+q_{m_2}+m_2+1}) = 7 \times 1 \times 2^{r_0+1-i}$; for $1 \le i \le r_0 + 1$,
where r_0 is greatest integer such that $7 \times 1 \times 2^{r_0} \le n + 2$,
$f(v_{i+p_0+p_1+...+p_{m_1}+m_1+q_0+q_1+...+q_{m_2}+m_2+1+r_0+1}) = 7 \times 3 \times 2^{r_1+1-i}$; for $1 \le i \le r_1 + 1$,
where r_1 is greatest integer such that $7 \times 3 \times 2^{r_1} \le n + 2$.

Continuing in this way till we get $\left\lfloor \frac{n+1}{2} \right\rfloor$ edges with label 1.

Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).

In view of above defined labeling pattern, we have $e_f(0) = \left\lceil \frac{n+1}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{n+1}{2} \right\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1$.

Case 2: Duplication of any edge other than e_1 and e_{n-1}.

Then $|V(G)| = n+2$ and $|E(G)| = n+2$.

$f(v'') = 1,$
$f(v_i) = 1 \times 1 \times 2^{p_0+1-i};$ for $1 \leq i \leq p_0,$
where p_0 is greatest integer such that $2^{p_0} \leq n+2,$
$f(v_{i+p_0}) = 3 \times 1 \times 2^{p_1+1-i};$ for $1 \leq i \leq p_1+1,$
where p_1 is greatest integer such that $3 \times 2^{p_1} \leq n+2,$
$f(v_{i+p_0+p_1+1}) = 3 \times 3 \times 2^{p_2+1-i};$ for $1 \leq i \leq p_2+1,$
where p_2 is greatest integer such that $3 \times 3 \times 2^{p_2} \leq n+2,$
$f(v_{i+p_0+p_1+p_2+2}) = 3 \times 3^2 \times 2^{p_3+1-i};$ for $1 \leq i \leq p_3+1,$
where p_3 is greatest integer such that $3 \times 3^2 \times 2^{p_3} \leq n+2,$
$f(v_{i+p_0+p_1+p_2+p_3+3}) = 3 \times 3^3 \times 2^{p_4+1-i};$ for $1 \leq i \leq p_4+1,$
where p_4 is greatest integer such that $3 \times 3^4 \times 2^{p_4} \leq n+2,$

Continuing for m_1 terms such that m_1 is greatest integer satisfying $3^{m_1} \leq n+2$ or we get $\left\lfloor \frac{n+2}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1}) = 5 \times 1 \times 2^{q_0+1-i};$ for $1 \leq i \leq q_0+1,$
where q_0 is greatest integer such that $5 \times 1 \times 2^{q_0} \leq n+2,$
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+1}) = 5 \times 3 \times 2^{q_1+1-i};$ for $1 \leq i \leq q_1+1,$
where q_1 is greatest integer such that $5 \times 3 \times 2^{q_1} \leq n+2,$
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+2}) = 5 \times 3^2 \times 2^{q_2+1-i};$ for $1 \leq i \leq q_2+1,$
where q_2 is greatest integer such that $5 \times 3^2 \times 2^{q_2} \leq n+2,$
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+q_2+3}) = 5 \times 3^3 \times 2^{q_3+1-i};$ for $1 \leq i \leq q_3+1,$
where q_3 is greatest integer such that $5 \times 3^3 \times 2^{q_3} \leq n+2,$

Continuing for m_2 terms such that m_2 is greatest integer satisfying $5 \times 3^{m_2} \leq n+2$ or we get $\left\lfloor \frac{n+2}{2} \right\rfloor$ edges with label 1,

$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1}) = 7 \times 1 \times 2^{r_0+1-i}$; for $1 \leq i \leq r_0+1$,
where r_0 is greatest integer such that $7 \times 1 \times 2^{r_0} \leq n+2$,
$f(v_{i+p_0+p_1+\ldots+p_{m_1}+m_1+q_0+q_1+\ldots+q_{m_2}+m_2+1+r_0+1}) = 7 \times 3 \times 2^{r_1+1-i}$; for $1 \leq i \leq r_1+1$,
where r_1 is greatest integer such that $7 \times 3 \times 2^{r_1} \leq n+2$.

Continuing in this way till we get $\left\lfloor \frac{n+2}{2} \right\rfloor$ edges with label 1.

Now label the remaining vertices in such a way that they neither divide nor divided by the label of adjacent vertices (except when one of the vertex label is 1).

In view of above defined labeling pattern, we have $e_f(0) = \left\lceil \frac{n+2}{2} \right\rceil$ and $e_f(1) = \left\lfloor \frac{n+2}{2} \right\rfloor$. Thus, $|e_f(0) - e_f(1)| \leq 1$.

Hence, the graph obtained by duplication of an arbitrary edge by a new edge in path P_n is a divisor cordial graph.

Illustration 4.14. *The graph obtained by duplication of an edge e_1 by a new edge in path P_5 and its divisor cordial labeling is shown in Fig 11.*

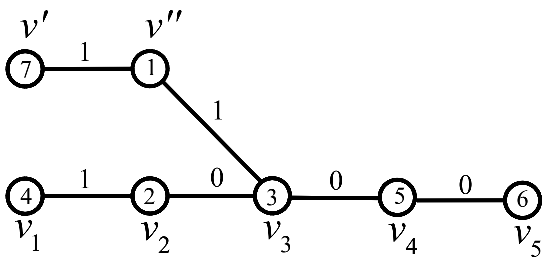

Fig 11: The graph obtained by duplication of an edge e_1 by a new edge in path P_5 and its divisor cordial labeling.

Illustration 4.15. *The graph obtained by duplication of an edge e_3 by a new edge in path P_6 and its divisor cordial labeling is shown in Fig 12.*

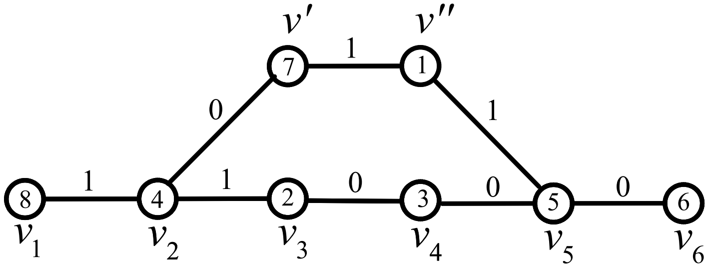

Fig 12: The graph obtained by duplication of an edge e_3 by a new edge in path P_6 and its divisor cordial labeling.

5. Conclusion

Divisor cordial labeling for ladders, triangular snakes and quadrilateral snakes is studied by Barasara and Thakkar [3, 4]. Vaidya and Shah [15] have discussed divisor cordial labeling for helms. Varatharajan *et al.* [16] have proved that paths are divisor cordial. While, In this paper, we have investigated divisor cordial labeling for larger graphs obtained from path, ladder, helm and snakes using some graph operations.

Acknowledgement

The second author is supported by Knowledge Consortium of Gujarat, Government of Gujarat, Ahmedabad through SHODH Scholarship – 2021-23 with Ref. No.: 202001400029.

References

[1] Barasara, C. M. and Thakkar, Y. B., Divisor cordial labeling for some cycle and wheel related graphs, Malaya Journal of Matematik, 8 (3) (2020), 966-972.

[2] Barasara, C. M. and Thakkar, Y. B., Divisor cordial labeling of graphs in the context of some graph operations, Advances and Applications in Discrete Mathematics, 28 (1) (2021), 31-48.

[3] Barasara, C. M. and Thakkar, Y. B., Divisor cordial labeling for ladders and total graph of some graphs, Advances and Applications in Mathematical Sciences, 21 (7) (2022), 3577-3594.

[4] Barasara, C. M. and Thakkar, Y. B., Divisor Cordial Labeling for Some Snakes and Degree Splitting Related Graphs, Communicated.

[5] Bosmia, M. I. and Kanani, K. K., Divisor cordial labeling in context of corona product, in 9th National Level Science Symposium, February 14, 2016, Organized by Christ College, Rajkot, Sponsored by GUJCOST, Gandhinagar, Mathematics and Statistics, 3 (2016), 178-182.

[6] Bosmia, M. I. and Kanani, K. K., Divisor cordial labeling in context of graph operations on bistar, Global J. Pure and Appl. Math., 12 (3) (2016), 2605-2618.

[7] Cahit, I., Cordial Graphs: A weaker version of graceful and harmonious Graphs, Ars Combinatoria, 23 (1987), 201-207.

[8] Clark, J. and Holton, D. A., A First Look at Graph Theory, World Scientific, London, 1991.

[9] Gallian, J. A., A Dynamic Survey of Graph Labeling, The Electronics Journal of Combinatorics, 24 (2021), #DS6.

[10] Murugan, A. N. and Devakiruba, G., Cycle Related Divisor Cordial Graphs, Int. J. Math. Trends Tech., 12 (1) (2014), 34-43.

[11] Prajapati, U. M. and Patel, P. A., Divisor Cordial Labeling in the Context of Friendship Graph, Journal of Xidian University, 14 (5) (2020), 167-177.

[12] Rokad, A. H. and Ghodasara, G. V., Divisor Cordial Labeling of Cycle Related Graphs, Int. J. for Research in Appl. Sci. and Eng. Tech., 3 (X) (2015), 341-346.

[13] Thirusangu, K. and Madhu, M., Divisor Cordial Labeling in Extended duplicate Graph of Star, Bistar and Double Star, J. Appl. Sci. and Compu., VI (I) (2019), 583-594.

[14] Vaidya, S. K. and Shah, N. H., Some star and bistar related divisor cordial graphs, Annals Pure Appl. Math., 3 (1) (2013), 67-77.

[15] Vaidya, S. K. and Shah, N. H., Further results on divisor cordial labeling, Annals Pure Appl. Math., 4 (2) (2013), 150-159.

[16] Varatharajan, R., Navanaeethakrishnan, S. and Nagarajan, K., Divisor Cordial Graphs, Int. J. of Mathematics and Combinatorics, 4 (2011), 15-25.

[17] Varatharajan, R., Navanaeethakrishnan, S. and Nagarajan, K., Special Classes of Divisor Cordial Graphs, Int. Mathematical Forum, 7 (35) (2012), 1737-1749.